

# Two-level lot-sizing with inventory bounds

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**Abstract:** We study a two-level uncapacitated lot-sizing problem with inventory bounds that occurs in a supply chain composed of a supplier and a retailer. The first level with the demands is the retailer level and the second one is the supplier level. The aim is to minimize the cost of the supply chain so as to satisfy the demands when the quantity of item that can be held in inventory at each period is limited. The inventory bounds can be imposed at the retailer level, at the supplier level or at both levels. We propose a polynomial dynamic programming algorithm to solve this problem when the inventory bounds are set on the retailer level. When the inventory bounds are set on the supplier level, we show that the problem is NP-hard. We give a pseudo-polynomial algorithm which solves this problem when there are inventory bounds on both levels. In the case where demand lot-splitting is not allowed, *i.e.* each demand has to be satisfied by a single order, we prove that the uncapacitated lot-sizing problem with inventory bounds is strongly NP-hard. This implies that the two-level lot-sizing problems with inventory bounds are also strongly NP-hard when demand lot-splitting is considered.

*Keywords:* Dynamic lot-sizing; inventory bounds; NP-hardness; dynamic programming.

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## 1 Introduction

We consider a two-level supply chain with a supplier and a retailer. The retailer has to satisfy a demand for a single item over a finite planning horizon of  $T$  periods. In order to satisfy the demand, the retailer has to determine a replenishment plan over the horizon, *i.e.* when and how many units to order. In order to satisfy the retailer's replenishment plan, the supplier has to determine a production plan. Ordering units induce a fixed ordering cost and a unit ordering cost for both actors. Carrying units in the inventory induce a unit holding cost for both actors as well. Moreover, the quantity that can be held in inventory at each period can be limited, since inventory bounds can be imposed at the retailer level, at the supplier level or at both levels. The cost of the supply chain is given by the sum of the supplier and the retailer total costs. The two-level Uncapacitated Lot-Sizing problem with Inventory Bounds consists in determining the order and the inventory quantities at each period for both replenishment and production plans in order to satisfy the external demand while minimizing the total cost of the supply chain.

### Literature review

For many practical applications, it is unreasonable to suppose that the inventory capacity is unlimited. In particular, the products that need temperature control or special storage facilities may have a limited storage capacity. This is for example the case in the pharmaceutical industry [2]. These constraints have led to the study of lot-sizing problems with inventory bounds.

The single level Uncapacitated Lot-Sizing problem with Inventory Bounds (ULS-IB) was first introduced by Love [7]. He proves that the problem with piecewise concave ordering and holding costs and backlogging can be solved using an  $\mathcal{O}(T^3)$  dynamic programming algorithm. More recently, Hwang *et al.* [4] propose an  $\mathcal{O}(T^2)$  algorithm based on geometric arguments. Atamtürk *et al.* [2] study the ULS-IB problem under the cost structure assumed in Love's paper [7], considering in addition a fixed holding cost. They propose an  $\mathcal{O}(T^2)$  algorithm to solve the problem. They also make a polyhedral study of the ULS-IB problem [1] by considering two cost structures: linear holding costs, linear and fixed holding costs. They provide an exact

separation algorithm for each problem. Other authors provide algorithms with better time complexity to solve this problem with particular costs structures [6, 10].

The multi-level uncapacitated lot-sizing problem can be solved with an  $\mathcal{O}(T^3)$  dynamic programming algorithm proposed by Zangwill [11]. More recently, Melo and Wolsey [8] improve the time complexity for solving the two-level case by proposing an  $\mathcal{O}(T^2 \log T)$  dynamic programming algorithm. A few papers deal with the 2ULS problem with inventory bounds. Jaruphongsa *et al.* [5] study this problem with demand time window constraints and stationary inventory bounds at the supplier level. They consider specific costs that make this problem solvable in  $\mathcal{O}(T^3)$  using a dynamic programming algorithm. They also prove that under the no lot-splitting assumption, *i.e.* each demand is satisfied by a single order, the problem is NP-hard.

## Contributions

The main contributions of the paper consist in setting the complexity of single-item 2ULS problems with inventory bounds. We consider that either the supplier, the retailer, or both of them, have a limited inventory capacity. A polynomial dynamic programming algorithm is provided to solve the problem with inventory bounds at the retailer level. The problem is shown to be weakly NP-hard when the inventory bounds are imposed at the supplier level. A complexity analysis for this class of problem is also proposed under the no lot-splitting assumption. In the sequel, we will denote 2ULS-IB<sub>R</sub> (resp. 2ULS-IB<sub>S</sub>), the problem where at each period, the inventory quantity at the retailer (resp. supplier) level cannot exceed the inventory bound. Finally, the 2ULS-IB<sub>SR</sub> problem is the problem where both the supplier and the retailer have a limited inventory capacity.

This paper is organized as follows. A mathematical formulation for the single-item 2ULS problem with inventory bounds is provided in Section 2. In Section 3, we propose a polynomial algorithm to solve the 2ULS-IB<sub>R</sub> problem. In Section 4, we prove that the 2ULS-IB<sub>S</sub> problem is NP-hard. We also show that the 2ULS-IB<sub>SR</sub> problem is solvable using a pseudo-polynomial algorithm in Section 5. Finally, we prove in Section 6 that these problems are strongly NP-hard under the no lot-splitting assumption where each demand has to be satisfied by a unique order.

## 2 Mathematical formulations

In this section, we describe the mathematical formulation of the 2ULS problem as well as the inventory bound constraints for the addressed problems.

We denote by  $d_t$  the demand at each period  $t$  for  $t \in \{1, \dots, T\}$ . The retailer's (resp. supplier's) costs are defined by a fixed ordering cost  $f_t^R$  (resp.  $f_t^S$ ), a unit ordering cost  $p_t^R$  (resp.  $p_t^S$ ) and a unit holding cost  $h_t^R$  (resp.  $h_t^S$ ) for  $t \in \{1, \dots, T\}$ . The retailer's (resp. supplier's) inventory bound at each period  $t$  is denoted by  $u_t^R$  (resp.  $u_t^S$ ) for  $t \in \{1, \dots, T\}$ .

We denote by  $x_t^R$  (resp.  $x_t^S$ ) the quantity ordered by the retailer (resp. supplier) at period  $t$ ,  $s_t^R$  (resp.  $s_t^S$ ) the retailer's (resp. supplier's) inventory level at the end of period  $t$  and  $y_t^R$  (resp.  $y_t^S$ ) the retailer's (resp. supplier's) setup variable, which is equal to 1 if an order occurs at period  $t$  at the retailer (resp. supplier).

level or 0 otherwise. The 2ULS problem can be formulated as follows:

$$\min \sum_{t=1}^T (f_t^S y_t^S + p_t^S x_t^S + h_t^S s_t^S + f_t^R y_t^R + p_t^R x_t^R + h_t^R s_t^R) \quad (1)$$

$$\text{s.t. } s_{t-1}^R + x_t^R = d_t + s_t^R \quad \forall t \in \{1, \dots, T\}, \quad (2)$$

$$s_{t-1}^S + x_t^S = x_t^R + s_t^S \quad \forall t \in \{1, \dots, T\}, \quad (3)$$

$$x_t^R \leq M_t^R y_t^R \quad \forall t \in \{1, \dots, T\}, \quad (4)$$

$$x_t^S \leq M_t^S y_t^S \quad \forall t \in \{1, \dots, T\}, \quad (5)$$

$$x^S, x^R, s^S, s^R \geq 0$$

$$y^S, y^R \in \{0, 1\}^T$$

where  $M_t^R = M_t^S = \sum_{i=t}^T d_i$ .

The supply chain total cost is given by (1). Constraints (2) (resp. (3)) are the inventory balance constraints at the retailer (resp. supplier) level. The supplier demand is the amount ordered at the retailer level at each period  $t$ . Constraints (4) and (5) force the setup variables to be equal to 1 if there is an order, *i.e.* if  $x_t^R > 0$  or  $x_t^S > 0$  respectively.

The 2ULS problem can be viewed as a fixed charge network flow problem (see Figure 1) where the nodes represent the periods at each level. A dummy node is also considered. For each node, the vertical inflows are the ordering quantities and the horizontal outflows represent the inventory quantities. In addition, arcs representing the external demand at each period at the retailer level are considered. In the sequel, we will not represent the dummy node, and the arcs will be represented only if they are active (*i.e.* a vertical arc will be represented if the corresponding ordering quantity is positive, and a horizontal arc is represented if the corresponding inventory quantity is not null).

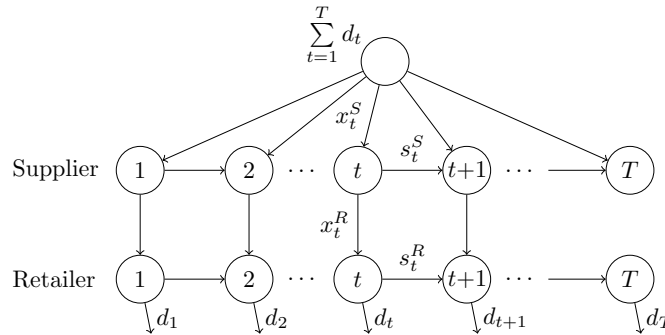


Figure 1: The 2ULS problem as a fixed charge network flow.

In addition to this classical problem, we introduce inventory bounds constraints. The inventory bounds constraints for the 2ULS-IB<sub>R</sub> problem are given by:

$$s_t^R \leq u_t^R \quad \forall t \in \{1, \dots, T\} \quad (6)$$

The mathematical formulation can be strengthened by setting  $M_t^R$  to  $\min(d_t + u_t^R, \sum_{i=t}^T d_i)$  in Constraint (4). Similarly, the inventory bounds constraints for the 2ULS-IB<sub>S</sub> problem are given by:

$$s_t^S \leq u_t^S \quad \forall t \in \{1, \dots, T\} \quad (7)$$

Similarly, parameter  $M_t^S$  can be replaced by  $\min(d_t + u_t^S + u_t^R, \sum_{i=t}^T d_i)$  in Constraint (5). The mathematical formulation of the 2ULS-IB<sub>SR</sub> problem is obtained by adding Constraints (6) and (7) to the mathematical formulation of the 2ULS problem.

### 3 Solving the 2ULS-IB<sub>R</sub> problem

In this section, we consider the 2ULS-IB<sub>R</sub> problem. To the best of our knowledge, this problem has not been studied in the literature yet. We present some structural properties of an optimal solution of the problem and propose an  $\mathcal{O}(T^4)$  algorithm to solve it. Since the inventory bounds are only set at the retailer level, the superscript  $R$  will be omitted in the inventory bound parameter  $u_t^R$  that will be denoted by  $u_t$ .

We recall that there exists an optimal solution to the 2ULS problem that verifies the Zero Inventory Ordering (ZIO) property [11], *i.e.*  $s_{t-1}^i x_t^i = 0$  for all  $t \in \{1, \dots, T\}$  and  $i \in \{S, R\}$ . This means that a period  $t$  is an ordering period only when the inventory quantity at period  $t - 1$  is zero.

Thereafter, we will use the following observation introduced in [9].

**Observation 1** *We can assume that  $u_{t-1} \leq u_t + d_t$  for all  $t \in \{1, \dots, T\}$ .*

Indeed, an optimal solution of the 2ULS-IB<sub>R</sub> problem where the inequality is not imposed satisfies  $\tilde{s}_{t-1}^R \leq u_t + d_t$  for all  $t \in \{1, \dots, T\}$ , where  $\tilde{s}^R$  is the optimal inventory plan of the retailer. This implies that the solution remains optimal if we assume that  $u_{t-1} \leq u_t + d_t$  for all  $t \in \{1, \dots, T\}$ . If this is not the case, *i.e.* if there exist a period  $t - 1$  such that  $u_{t-1} > u_t + d_t$ , then we can set  $u_{t-1} = u_t + d_t$ . An optimal solution of the 2ULS-IB<sub>R</sub> problem on this modified instance will be an optimal solution of the 2ULS-IB<sub>R</sub> problem in the original instance.

#### 3.1 Dominance properties

In this section, we propose some dominance properties in order to determine an efficient solving approach.

We know that the ZIO property does not hold for the ULS-IB problem [6, 7]. Let us first show that for the 2ULS-IB<sub>R</sub> problem, the cost of the best solution in which the ZIO property is fulfilled at the retailer level may be arbitrarily large compared to the cost of an optimal solution in which the ZIO property is not required.

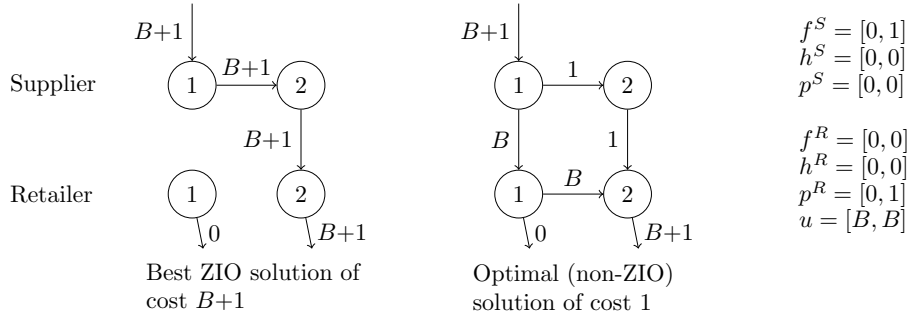
**Property 1** *For the 2ULS-IB<sub>R</sub> problem, the cost of the best ZIO policy at the retailer level may be arbitrarily large compared to the cost of an optimal policy.*

**Proof.** Consider the following instance  $\mathcal{I}$ :  $T = 2$ ,  $h^S = p^S = f^R = h^R = [0, 0]$ ,  $f^S = [0, 1]$ ,  $p^R = [0, 1]$ ,  $d = [0, B + 1]$  and  $u = [B, B]$ , where  $B$  is a large constant. The best solution satisfying the ZIO property at the retailer level is given by  $x^S = [d_2, 0]$ ,  $x^R = [0, d_2]$ . The corresponding cost is  $B + 1$  whereas the optimal non-ZIO solution is given by  $x^S = [d_2, 0]$ ,  $x^R = [B, 1]$  inducing a cost equals to 1 (see Figure 2).  $\square$

Let us now give the definition of a block (Definition 3), previously introduced in [2, 1] for the single level case. This definition will allow us to characterize an optimal solution for the 2ULS-IB<sub>R</sub> problem.

**Definition 1 (Subplan)** *Let  $i$  and  $j$  be two periods such that  $1 \leq i \leq j \leq T$ . A subplan  $[i, j]$  is a partial solution at the retailer level of the 2ULS-IB<sub>R</sub> problem between the periods  $i$  and  $j$  defined by  $x_i^R, \dots, x_j^R$ .*

**Definition 2 (Regular subplan)** *Let  $i$  and  $j$  be two periods such that  $1 \leq i \leq j \leq T$ . A regular subplan  $[i, j]$  is a subplan  $[i, j]$  such that  $s_{i-1}^R \in \{0, u_{i-1}\}$  and  $s_j^R \in \{0, u_j\}$ .*

Figure 2: Solutions for the instance  $\mathcal{I}$  of the 2ULS-IB<sub>R</sub> problem

**Definition 3 (Block)** Let  $i$  and  $j$  be two periods such that  $1 \leq i \leq j \leq T$ . Let  $\alpha \in \{0, u_{i-1}\}$  and  $\beta \in \{0, u_j\}$ . A block  $[i, j]_\beta^\alpha$  is a regular subplan  $[i, j]$  where  $s_{i-1}^R = \alpha$ ,  $s_j^R = \beta$  and  $0 < s_t^R < u_t$  for all  $t \in \{i, \dots, j-1\}$ .

In other words,  $[i, j]_\beta^\alpha$  is a block at the retailer level if the inventory quantity at the beginning of period  $i$  (denoted  $\alpha$ ) is either 0 or  $u_{i-1}$ , the inventory quantity at the end of period  $j$  (denoted  $\beta$ ) is either 0 or  $u_j$ , and the inventory quantities for each period between  $i$  and  $j$  is strictly positive and not equal to the inventory bound. A regular subplan is thus made of one or several blocks.

**Definition 4 (Order quantity)** Let  $d_{tk} = \sum_{i=t}^k d_i$  be the cumulative demand between periods  $t$  and  $k$ . The order quantity at the retailer level in a subplan  $[i, j]$  is given by  $X_{ij} = d_{ij} - s_{i-1}^R + s_j^R$ .

Observe that for a block  $[i, j]_\beta^\alpha$ ,  $X_{ij} = d_{ij} - \alpha + \beta$ . Thereafter, we give some properties observed by an optimal solution for the 2ULS-IB<sub>R</sub> problem.

**Property 2** There exists an optimal solution of the 2ULS-IB<sub>R</sub> problem where the following properties hold at the retailer level for all  $1 \leq i \leq j \leq T$ ,  $\alpha \in \{0, u_{i-1}\}$ , and  $\beta \in \{0, u_j\}$ :

- (i) If  $[i, j]_\beta^\alpha$  is a block, then there is at most one ordering period in this block.
- (ii) If  $[i, j]_\beta^0$  is a block and if there is an ordering period in this block, then this ordering period is  $i$ .
- (iii) If  $[i, j]_{u_j}^\alpha$  is a block and if there is an ordering period in this block, then this ordering period is  $j$ .

**Proof.** (i) Let us assume that there exists two consecutive ordering periods  $k_1$  and  $k_2$  in a block  $[i, j]_\beta^\alpha$ ,  $i \leq k_1 < k_2 \leq j$ , of an optimal solution  $(\tilde{y}^R, \tilde{x}^R, \tilde{s}^R)$ . We prove that an optimal solution  $(y^R, x^R, s^R)$  can be derived from this solution so that at most one ordering period occurs in the block  $[i, j]_\beta^\alpha$  of this new solution. Starting from  $(\tilde{y}^R, \tilde{x}^R, \tilde{s}^R)$ , we iteratively transfer one unit ordered at period  $k_2$  to period  $k_1$  until one of the following cases occur:

Case 1: each unit that is ordered at period  $k_2$  is then ordered at period  $k_1$  and the property  $0 < s_t^R < u_t$  still holds for all  $t \in \{i, \dots, j-1\}$ . Thus,  $k_2$  is not an ordering period anymore.

Case 2: there exists a period  $t$  such that  $k_1 \leq t < k_2$  and  $s_t^R = u_t$ . This implies that the block  $[i, j]_\beta^\alpha$  will be decomposed into two blocks defined by  $[i, t]_{u_t}^\alpha$  and  $[t+1, j]_\beta^{u_t}$  in  $(y^R, x^R, s^R)$ . Using the same arguments, we can show that each of these blocks can be decomposed into blocks with at most one ordering period.

The cost of the solution  $(y^R, x^R, s^R)$  is equal to the cost of the optimal solution  $(\tilde{y}^R, \tilde{x}^R, \tilde{s}^R)$ . Indeed, if the cost of transferring one unit from period  $k_2$  to period  $k_1$  was greater than the cost of ordering one unit at period  $k_2$ , then it would contradict the optimality of the solution  $(\tilde{y}^R, \tilde{x}^R, \tilde{s}^R)$ .

(ii) If  $d_i > 0$ , then period  $i$  is necessarily an ordering period since  $s_{i-1}^R = 0$ .

If  $d_i = 0$  and there is an ordering period in the block  $[i, j]_\beta^0$ , then we have  $s_i^R > 0$  and period  $i$  is necessarily an ordering period since  $s_{i-1}^R = 0$ .

(iii) Assume that the (unique) ordering period is  $k$  in the block  $[i, j]_{u_j}^\alpha$  with  $i \leq k \leq j-1$ . From Observation 1, we have  $u_k \leq u_{k+1} + d_{k+1} \leq u_{k+2} + d_{k+2} + d_{k+1} \leq \dots \leq u_j + d_j + \sum_{i=k+1}^{j-1} d_i$ . Thus  $u_k \leq u_j + d_{k+1,j}$ . There are two possible cases:

Case 1:  $u_k = u_j + d_{k+1,j}$ . In this case, since  $s_j^R = u_j$ , then  $s_k^R = u_k$  which is not possible since  $[i, j]_{u_j}^\alpha$  is a block.

Case 2:  $u_{j-1} < u_j + d_j$ . In this case, it is not possible to have  $s_j^R = u_j$  without having an additional ordering period in the block, which contradicts Property 2. So, the ordering period has to be at period  $j$  in a block  $[i, j]_{u_j}^\alpha$ .  $\square$

Since the storage capacity is not limited at the supplier level for the 2ULS-IB<sub>R</sub> problem, we have the following dominance property:

**Property 3** *The ZIO property holds at the supplier level for the 2ULS-IB<sub>R</sub> problem.*

Using these properties, we propose a polynomial algorithm to solve the 2ULS-IB<sub>R</sub> problem.

### 3.2 A polynomial dynamic programming algorithm

In this section, we derive a polynomial backward dynamic programming algorithm to solve the 2ULS-IB<sub>R</sub> problem. The rationale of this algorithm is to compute a block decomposition of the retailer's replenishment plan such that the total cost of the supply chain is minimized using the dominance properties of the optimal solutions of the problem.

#### 3.2.1 Recursion formula

Let  $i, j$  be two periods such that  $1 \leq i \leq j \leq T$ . Let us consider a regular subplan  $[i, j]$  of a solution of the 2ULS-IB<sub>R</sub> problem. Notice that by definition  $[i, j]$  is not necessarily a block unless property  $0 < s_k^R < u_k$  for all  $k \in \{i, \dots, j-1\}$  holds. Assume that at period  $t$ , an order quantity  $X_{ij} = d_{ij} - s_{i-1}^R + s_j^R$  (see Definition 4) is available at the supplier level, *i.e.* it is either ordered at period  $t$  or stored at period  $t-1$  assuming the ZIO policy. The aim is to decompose the regular subplan  $[i, j]$  into blocks satisfying Property 2. Indeed, a solution of the 2ULS-IB<sub>R</sub> problem is a regular subplan  $[1, T]$ , and a regular subplan is a sequence of blocks.

An example is given in Figure 3. The graph represents subplans of a solution for an instance of the 2ULS-IB<sub>R</sub> problem where  $T = 4$ . At period  $t = 1$ , a quantity  $X_{11} = d_1 + u_1$  is available and at period  $t = 2$ , a quantity  $X_{24} = d_{24} - u_1$  is available at the supplier level (it is also available at period 3). In this example,  $[2, 4]$  is a regular subplan composed of the blocks  $[2, 3]_{u_1}^0$  and  $[4, 4]_0^0$ .

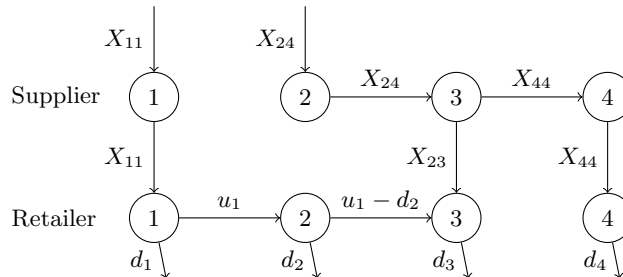


Figure 3: Subplans decomposition for an instance of the 2ULS-IB<sub>R</sub> problem where  $T = 4$

### Determining the retailer's orders in a regular subplan $[i, j]$ .

We assume that a quantity  $X_{ij}$  of a regular subplan  $[i, j]$  is available at period  $t$  at the supplier level. In order to define  $G_{tij}^{\alpha\beta}$ , the optimal cost to cover the demands  $d_{ij}$  of the subplan  $[i, j]$  such that  $s_{i-1}^R = \alpha$  and  $s_j^R = \beta$ , we first need to introduce the optimal cost of a block and the cost of a regular subplan composed by at least two blocks.

*Definition of the cost  $\phi_{ijk}^{\alpha\beta}$  of a block  $[i, j]_\beta^\alpha$*

We consider a regular subplan  $[i, j]$  at the retailer level assuming that at most one order occurs at a given period  $k \in \{i, \dots, j\}$ , and where  $s_{i-1}^R = \alpha \in \{0, u_{i-1}\}$  and  $s_j^R = \beta \in \{0, u_j\}$ . In this subplan, either an order occurs at period  $k$  or the demands can be covered using the inventory quantities. If the inventory variables of the subplan  $[i, j]$  satisfy the block requirements, then  $[i, j]$  constitutes a block  $[i, j]_\beta^\alpha$ . Satisfying the demands of the block  $[i, j]_\beta^\alpha$  induces a cost that will be denoted  $\phi_{ijk}^{\alpha\beta}$ .

We will denote by  $\phi_{ij-}^{\alpha\beta}$  the cost of the block  $[i, j]_\beta^\alpha$  without ordering period.

*Definition of the cost  $v_{tij}^{\alpha\beta}$  of a regular subplan  $[i, j]$  composed of at least two blocks*

Let  $[i, j]$  be a regular subplan composed of at least two blocks such that  $s_{i-1}^R = \alpha$  and  $s_j^R = \beta$ . We denote by  $v_{tij}^{\alpha\beta}$  the optimal cost of  $[i, j]$  assuming that a quantity  $X_{ij}$  is available at period  $t$  at the supplier level,  $1 \leq t \leq j$ .

*Definition of the cost  $G_{tij}^{\alpha\beta}$*

We assume that a quantity  $X_{ij}$  of a regular subplan  $[i, j]$  is available at period  $t$  at the supplier level with  $1 \leq i \leq j \leq T$  and  $1 \leq t \leq j$ ,  $s_{i-1}^R = \alpha$  and  $s_j^R = \beta$ . Let  $G_{tij}^{\alpha\beta}$  be the optimal cost to cover the demands  $d_{ij}$  of the regular subplan  $[i, j]$ . The fixed ordering costs  $f^S$  at the supplier level are not included in  $G_{tij}^{\alpha\beta}$ . Computing  $G_{tij}^{\alpha\beta}$  is based on the cost computation of the blocks that compose the subplan  $[i, j]$ .

Two cases will be considered to compute  $G_{tij}^{\alpha\beta}$  depending on the ordering quantity  $X_{ij}$ . The cost  $G_{tij}^{\alpha\beta}$  is given by:

$$G_{tij}^{\alpha\beta} = \begin{cases} \min \left\{ \min_{t \leq k \leq j} \{ \phi_{ijk}^{\alpha\beta} + \sum_{l=t}^{k-1} h_l^S X_{ij} \}, v_{tij}^{\alpha\beta} \right\}, & \text{if } X_{ij} > 0 \\ \phi_{ij-}^{\alpha\beta}, & \text{if } X_{ij} = 0 \end{cases} \quad (8)$$

In Equation (8), the term  $\min_{t \leq k \leq j} \{ \phi_{ijk}^{\alpha\beta} + \sum_{l=t}^{k-1} h_l^S X_{ij} \}$  represents the optimal cost of the regular subplan  $[i, j]$  when it is made of a single block, and  $v_{tij}^{\alpha\beta}$  is the optimal cost of  $[i, j]$  when it is composed of at least two blocks.

The values  $\phi_{ijk}^{\alpha\beta}$  and  $G_{tij}^{\alpha\beta}$  will be computed by the dynamic programming algorithm. If  $[i, j]_\beta^\alpha$  is not a block or if  $k < i$  then the cost  $\phi_{ijk}^{\alpha\beta}$  will be equal to  $+\infty$ . Moreover,  $G_{tij}^{\alpha\beta}$  will be equal to  $+\infty$  if  $i > j$  or  $t > j$ .

### Finding the decomposition of a regular subplan $[i, j]$ into at least two blocks.

Let us see how to compute the cost  $v_{tij}^{\alpha\beta}$  of the regular subplan  $[i, j]$  that is composed of at least two blocks.

*Definition of the cost  $w_{tij}^{\alpha\gamma\beta}$  where  $k$  is the ordering period of the first block*

We assume that a quantity  $X_{ij}$  of a regular subplan  $[i, j]$  is available at period  $t$  at the supplier level. Let  $w_{tij}^{\alpha\gamma\beta}$  be the optimal cost of the subplan  $[i, j]$  where  $s_{i-1}^R = \alpha$ ,  $s_j^R = \beta$  and  $k$  is the ordering period of the first block  $[i, l]_\gamma^\alpha$  of  $[i, j]$ ,  $i \leq l < j$ ,  $\gamma \in \{0, u_l\}$ . The aim is to find a period  $l$  ( $k \leq l < j$ ) such that  $[i, l]_\gamma^\alpha$  is the first block of the regular subplan  $[i, j]$  with an order at period  $k$  in an optimal solution.

A representation of the structure of the regular subplan is given in Figure 4. There are  $X_{ij}$  units available

at period  $t$  at the supplier level. At the retailer level, the regular subplan  $[i, j]$  where  $s_{i-1}^R = \alpha$  and  $s_j^R = \beta$  is composed of a first block  $[i, l]_\gamma^\alpha$ . In order to satisfy the demands of this block, a quantity  $X_{il}$  is ordered at period  $k$  in this block. At the supplier level, a quantity  $X_{ij}$  is stored from period  $t$  to period  $k$ . After the retailer's order at period  $k$ , the inventory quantity is  $X_{l+1,j}$  at the supplier level which will be used to satisfy the demands of the subplan  $[l+1, j]$ .

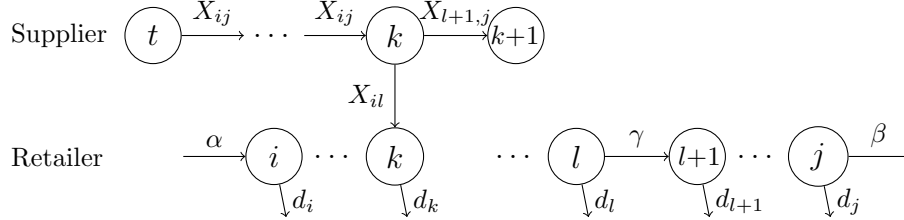


Figure 4: Illustration of the cost  $w_{tijk}^{\alpha\gamma\beta}$  where  $[i, j]$  is a regular subplan and  $X_{ij}$  is available at the supplier level at period  $t$ .

The cost  $w_{tijk}^{\alpha\gamma}$  is then given by:

$$w_{tijk}^{\alpha\gamma\beta} = \min_{k \leq l < j} \{ \phi_{ilk}^{\alpha\gamma} + G_{k,l+1,j}^{\gamma\beta} + \sum_{p=t}^{k-1} h_p^S X_{ij} \} \quad (9)$$

The first term  $\phi_{ilk}^{\alpha\gamma}$  in Equation (9) represents the cost of satisfying the demands of the block  $[i, l]_\gamma^\alpha$  with an ordering at period  $k$ . The second term  $G_{k,l+1,j}^{\gamma\beta}$  in Equation (9) represents the optimal cost for satisfying the demands of the regular subplan  $[l+1, j]$  assuming that the quantity  $X_{l+1,j}$  is available at period  $k$  at the supplier level. Finally, the last term  $\sum_{p=t}^{k-1} h_p^S X_{ij}$  represents the cost of carrying  $X_{ij}$  units from period  $t$  to period  $k$  at the supplier level.

In the sequel, we denote by  $l^* = \operatorname{argmin} w_{tijk}^{\alpha\gamma\beta}$ . The first block of the regular subplan  $[i, j]$  is  $[i, l^*]_\gamma^\alpha$  and the ordering period in this block is  $k$  if it exists. The cost  $v_{tij}^{\alpha\beta}$  is given by:

$$v_{tij}^{\alpha\beta} = \min_{i \leq k < j; \gamma \in \{0, u_{l^*}\}} \{ w_{tijk}^{\alpha\gamma\beta} \} \quad (10)$$

### Computing the cost of a block $[i, j]_\beta^\alpha$ .

Finally, let us see how to compute the cost  $\phi_{ijk}^{\alpha\beta}$  of a block  $[i, j]_\beta^\alpha$  where  $k$  is an ordering period if it exists. We distinguish several cases.

If we have a block  $[i, j]_\beta^0$  with an ordering period  $k$ , since the inventory at the beginning of period  $i$  is null, an order must occur at period  $k = i$  (Property 2). The cost  $\phi_{ijk}^{0\beta}$  is then given by:

$$\phi_{ijk}^{0\beta} = \begin{cases} f_i^R + p_i^R X_{ij} + \sum_{n=i}^j h_n^R (d_{n+1,j} + \beta), & \text{if } k = i \text{ and } 0 < d_{ij} + \beta \leq u_i \\ 0, & \text{if } i = j \text{ and } d_{ij} + \beta = 0 \\ +\infty, & \text{otherwise} \end{cases}$$

The cost  $\phi_{ijk}^{0\beta}$  is equal to  $+\infty$  if  $[i, j]_\beta^0$  is not a block.



If  $[i, j]_{u_j}^{u_{i-1}}$  is a block such that  $u_{i-1} = d_{ij} + u_j$ , then the demands of this block can be satisfied without setting any order between period  $i$  and  $j$  since  $s_{i-1}^R = u_{i-1}$ . Thus, the cost  $\phi_{ijk}^{u_{i-1}u_j}$  of the block is given by:

$$\phi_{ijk}^{u_{i-1}u_j} = \sum_{n=i}^j h_n^R(u_{i-1} - d_{in} + u_j), \text{ if } u_{i-1} = d_{ij} + u_j$$

Otherwise, if we have a block  $[i, j]_{u_j}^{u_{i-1}}$  and  $u_{i-1} < d_{ij} + u_j$ , then a quantity  $X_{ij}$  has to be ordered. Since the inventory quantity at the end of period  $j$  is  $u_j$ , the ordering period is  $j$  (Property 2). Moreover, we have to ensure that the inventory bounds constraints are not violated, and that the demands  $d_{i,j-1}$  can be covered by the inventory quantity at the end of period  $i-1$ , i.e.  $u_{i-1} > d_{i,j-1}$ . In this case, the cost  $\phi_{ijk}^{u_{i-1}u_j}$  is given by:

$$\begin{aligned} \phi_{ijk}^{u_{i-1}u_j} &= f_k^R + p_k^R X_{ij} + \sum_{n=i}^{j-1} h_n^R(u_{i-1} - d_{in}) + h_j^R u_j, \\ &\text{if } k = j \text{ and } d_{ij} + u_j > u_{i-1} > d_{i,j-1} \end{aligned}$$

If  $[i, j]_{u_j}^{u_{i-1}}$  is not a block, then the cost  $\phi_{ijk}^{u_{i-1}u_j}$  is equal to  $+\infty$ .

If we have a block  $[i, j]_0^{u_{i-1}}$  and  $u_{i-1} < d_{ij}$ , then at most one order can occur at period  $k$  (Property 2). When the inventory quantity at the end of period  $j$  is null, the quantity  $X_{ij}$  can be ordered at any period  $k$  between  $i$  and  $j$ . In this case, we have to ensure that the inventory bounds constraints are not violated, that the inventory quantity  $u_{i-1}$  covers the demands before period  $k$  ( $u_{i-1} > d_{i,k-1}$ ), and that the demands after period  $k$  can be satisfied ( $u_k \geq d_{k+1,j}$ ). The cost  $\phi_{ijk}^{u_{i-1}0}$  is then given by:

$$\begin{aligned} \phi_{ijk}^{u_{i-1}0} &= f_k^R + p_k^R X_{ij} + \sum_{n=i}^{k-1} h_n^R(u_{i-1} - d_{in}) + \sum_{n=k}^j h_n^R d_{n+1,j}, \\ &\text{if } d_{ij} > u_{i-1} > d_{i,k-1} \text{ and } u_k \geq d_{k+1,j} \end{aligned}$$

### Determining the supplier's orders.

The aim is to determine the ordering periods at the supplier level in order to satisfy the demands of a regular subplan. Since the ZIO property holds at the supplier level (Property 3), each retailer's order quantity at a given period is entirely produced in a single period at the supplier level.

Let  $C_{tij}^{\alpha\beta}$  be the optimal cost of satisfying the demands  $d_{ij}$  of the regular subplan  $[i, j]$  where the first ordering period at the supplier level is larger than or equal to  $t$ , with  $1 \leq i \leq j \leq T$ ,  $1 \leq t \leq j$ ,  $\alpha \in \{0, u_{i-1}\}$  and  $\beta \in \{0, u_j\}$ . The total ordering quantity of the subplan is equal to  $X_{ij}$ .

If  $X_{ij} > 0$ , then the quantity  $X_{ij}$  is completely or partially ordered at period  $t$  or at a subsequent period if no order occurs at period  $t$  at the supplier level. The cost  $C_{tij}^{\alpha\beta}$  is then given by the following equation where  $\mathbb{K}(x)$  is a function equals to 0 if  $x = 0$  and  $+\infty$  otherwise.

$$\begin{aligned} C_{tij}^{\alpha\beta} &= \min \left\{ C_{t+1,i,j}^{\alpha\beta}, f_t^S + p_t^S X_{ij} + G_{tij}^{\alpha\beta}, \right. \\ &\quad \left. \min_{i \leq l < j; \gamma \in \{0, u_l\}} \{ \min(f_t^S + p_t^S X_{il}, \mathbb{K}(X_{il})) + G_{til}^{\alpha\gamma} + C_{t^*+1,l+1,j}^{\gamma\beta} \} \right\} \end{aligned} \quad (11)$$

where  $t^*$  is the last ordering period at the retailer level in the regular subplan  $[i, l]$  ( $t^*$  is determined and stored when the cost  $G_{til}^{\alpha\gamma}$  is computed). The period  $t^*$  is the earliest ordering period from which the supplier can order for satisfying the demands of the regular subplan  $[l+1, j]$ . If there is no ordering period

in the regular subplan  $[i, l]$ , then we set  $t^* = t$ .

The first term in Equation (11) corresponds to the case where there is no order at period  $t$  at the supplier level. The second term in Equation (11) corresponds to the case where a quantity  $X_{ij}$  is ordered at period  $t$  at the supplier level. Finally, the last term in Equation (11) represents the case where the quantity  $X_{ij}$  is partially ordered at period  $t$  at the supplier level: a quantity  $X_{il} > 0$  is ordered at period  $t$  to satisfy the demands of the regular subplan  $[i, l]$  with  $i \leq l < j$ . Because of the ZIO property at the supplier level, the supplier orders the quantity  $X_{l+1,j}$  after period  $t^*$ .

If  $X_{ij} = 0$ , then no order is required at the supplier level. The cost is then equal to the cost  $G_{tij}^{\alpha\beta}$  of the subplan  $[i, j]$  with  $s_{i-1}^R = \alpha$  and  $s_j^R = \beta$ :

$$C_{tij}^{\alpha\beta} = G_{tij}^{\alpha\beta} \quad (12)$$

### Optimal cost.

The optimal cost of satisfying the demands of the regular subplan  $[1, T]$  is given by  $C_{11T}^{00}$  since  $s_0^R = 0$ ,  $s_T^R = 0$  and the earliest order period at the supplier level is  $t = 1$ .

### 3.2.2 Complexity analysis

A pre-processing phase will consist in the computation of  $d_{1j}$  for all  $j \in \{1, \dots, T\}$  in  $\mathcal{O}(T)$ . Therefore, each  $d_{ij}$  for all  $i, j \in \{1, \dots, T\}$  can be computed in constant time. Moreover, the holding costs required in the computation of each cost component is pre-computed and stored in  $\mathcal{O}(T^2)$ .

Therefore, the cost  $\phi_{ijk}^{\alpha\beta}$  can be computed and stored in  $\mathcal{O}(T^3)$  for all  $i, j, k \in \{1, \dots, T\}$ . Besides, it takes  $\mathcal{O}(T^4)$  time to compute and to store the costs  $G_{tij}^{\alpha\beta}$  and  $v_{tij}^{\alpha\beta}$ . Finally, the cost  $w_{tijk}^{\alpha\gamma\beta}$  is computed in  $\mathcal{O}(T^5)$ , and then the time complexity of the dynamic programming algorithm based on the recursion formula (11) to compute  $C_{11T}^{00}$  is  $\mathcal{O}(T^5)$ .

In what follows, we will show how the time complexity of computing the cost  $w_{tijk}^{\alpha\gamma\beta}$  can be improved from  $\mathcal{O}(T^5)$  to  $\mathcal{O}(T^4)$ . The observation below, provided by Atamtürk *et al.* [2] for the single level case, will be useful to compute the cost  $w_{tijk}^{\alpha\gamma\beta}$  more efficiently.

Observe that a regular subplan  $[i, j]$  with a single order at period  $k$  (with  $i \leq k \leq j$ ) does not always verify the block properties. In the case where  $[i, j]$  is not a block, then the regular subplan  $[i-1, j]$  with a single order at period  $k$ ,  $s_{i-2}^R = 0$  if  $s_{i-1}^R = 0$ ,  $s_{i-2}^R = u_{i-2}$  if  $s_{i-1}^R = u_{i-1}$  and  $s_j^R \in \{0, u_j\}$  is not a block. Moreover, because of Property 2, if we have a block  $[i, j]_\beta^0$  with an order at period  $k$ , then the regular subplan  $[i-1, j]$  with a single order at period  $k$ ,  $s_{i-2}^R = 0$  and  $s_j^R = \beta$  is not a block. Finally, if  $[i, j]$  is a block  $[i, j]_\beta^{u_{i-1}}$  with an order at period  $k$ , then the cost of the regular subplan  $[i-1, j]$  with a single order at period  $k$ ,  $s_{i-2}^R = u_{i-2}$  and  $s_j^R = \beta$  can be derived from the cost of  $[i, j]_\beta^{u_{i-1}}$  by considering the case where there is already an ordering period in  $[i, j]_\beta^{u_{i-1}}$  or not.

**Observation 2** For all  $1 \leq i \leq k \leq j \leq T$ ,  $\alpha \in \{0, u_{i-1}\}$  and  $\beta \in \{0, u_j\}$ , we have:

- (i) if  $\phi_{ijk}^{\alpha\beta} = +\infty$ , then  $\phi_{i-1,j,k}^{\alpha'\beta} = +\infty$  where  $\alpha' = 0$  if  $\alpha = 0$  and  $\alpha' = u_{i-2}$  if  $\alpha = u_{i-1}$ .
- (ii) if  $\phi_{ijk}^{0\beta} \neq +\infty$ , then  $\phi_{i-1,j,k}^{0\beta} = +\infty$
- (iii) if  $\phi_{ijk}^{u_{i-1}\beta} \neq +\infty$ , then:

$$\phi_{i-1,j,k}^{u_{i-2}\beta} = \begin{cases} \phi_{ijk}^{u_{i-1}\beta} + h_{i-2}^R u_{i-2} + (p_k^R - \sum_{l=i-1}^{k-1} h_l^R) \bar{X}_{i-1}, \\ \quad \text{if } u_{i-2} > d_{i-1,k-1}, u_{i-2} < u_{i-1} + d_{i-1} \text{ and } d_{ij} + \beta > u_{i-1} \\ \phi_{ijk}^{u_{i-1}\beta} + h_{i-2}^R u_{i-2} + f_k^R + (p_k^R - \sum_{l=i-1}^{k-1} h_l^R) \bar{X}_{i-1}, \\ \quad \text{if } u_{i-2} > d_{i-1,k-1}, u_{i-2} < u_{i-1} + d_{i-1} \text{ and } d_{ij} + \beta = u_{i-1} \\ +\infty, \text{ otherwise} \end{cases} \quad \text{where } \bar{X}_{i-1} = u_{i-1} - u_{i-2} + d_{i-1} \text{ is the quantity that has to be ordered in the block } [i-1, j]_{\beta}^{u_{i-2}} \text{ in addition to } X_{ij} = d_{ij} - u_{i-1} + \beta.$$

**Proof.** (i) If  $\phi_{ijk}^{\alpha\beta} = +\infty$ , then the regular subplan  $[i, j]$  with a single order at period  $k$ ,  $s_{i-1}^R = \alpha$  and  $s_j^R = \beta$  is not a block. The violation(s) observed in the regular subplan  $[i, j]$  will also hold for the regular subplan  $[i-1, j]$ .

(ii) If  $\phi_{ijk}^{0\beta} \neq +\infty$ , then  $[i, j]_{\beta}^0$  is a block, and by Property 2-(ii) there is an ordering period at  $k = i$ . We consider the regular subplan  $[i-1, j]$  with an order at period  $k$ ,  $s_{i-2}^R = 0$  and  $s_j^R = \beta$ . If  $d_{i-1} = 0$ , then the regular subplan  $[i-1, j]$  is not a block since  $s_{i-2}^R = 0$ . If  $d_{i-1} > 0$ , then  $d_{i-1}$  could not be covered and thus the regular subplan  $[i-1, j]$  is not a block.

(iii) If  $\phi_{ijk}^{u_{i-1}\beta} \neq +\infty$ , then  $[i, j]_{\beta}^{u_{i-1}}$  is a block with an ordering period  $k$  if it exists. We consider the regular subplan  $[i-1, j]$  with a single order at period  $k$ ,  $s_{i-2}^R = u_{i-2}$  and  $s_j^R = \beta$ . We want to determine if this regular subplan is a block.

We know that  $u_{i-2} \leq u_{i-1} + d_{i-1}$  (Observation 1). If  $u_{i-2} = u_{i-1} + d_{i-1}$ , then this regular subplan is not a block because in that case  $s_{i-1}^R = u_{i-1}$ . If  $u_{i-2} < u_{i-1} + d_{i-1}$ , then we have  $s_{i-1}^R < u_{i-1} \leq d_{ij} + \beta$ , and there must be an ordering period at  $k$  in the subplan  $[i-1, j]$ .

Moreover, if  $u_{i-2} > d_{i-1,k-1}$ , then we have a block  $[i-1, j]_{\beta}^{u_{i-2}}$ . The retailer has to order a quantity  $\bar{X}_{i-1} = u_{i-1} - u_{i-2} + d_{i-1} > 0$  in addition to  $X_{ij} = d_{ij} - u_{i-1} + \beta$  at period  $k$ . The inventory quantities between periods  $k$  and  $j$  remain unchanged in the block  $[i-1, j]_{\beta}^{u_{i-2}}$ . Since the demand  $d_{i-1}$  has to be covered by  $u_{i-2}$ , there are  $\bar{X}_{i-1}$  less units in the inventory between periods  $i-1$  and  $k-1$ . The cost  $\phi_{i-1,j,k}^{u_{i-2}\beta}$  of the block  $[i-1, j]_{\beta}^{u_{i-2}}$  can be derived from  $\phi_{ijk}^{u_{i-1}\beta}$  by considering these two cases:

Case 1: Assume that a quantity  $X_{ij} > 0$  is ordered at period  $k$  in the block  $[i, j]_{\beta}^{u_{i-1}}$ . Then, the cost of the block  $[i-1, j]_{\beta}^{u_{i-2}}$  is given by:  $\phi_{ijk}^{u_{i-1}\beta} + h_{i-2}^R u_{i-2} + (p_k^R - \sum_{l=i-1}^{k-1} h_l^R) \bar{X}_{i-1}$ .

Case 2: Assume that no ordering period occurs in the block  $[i, j]_{\beta}^{u_{i-1}}$ . Then, an additional fixed ordering cost  $f_k^R$  must be considered to compute the cost of the block  $[i-1, j]_{\beta}^{u_{i-2}}$ , which will be given by:  $\phi_{ijk}^{u_{i-1}\beta} + f_k^R + h_{i-2}^R u_{i-2} + (p_k^R - \sum_{l=i-1}^{k-1} h_l^R) \bar{X}_{i-1}$ .  $\square$

In a block  $[i-1, j]_{\beta}^{u_{i-2}}$ , we have that  $X_{i-1,j} = X_{ij} + \bar{X}_{i-1}$  where  $X_{ij}$  is the ordering quantity in a block  $[i, j]_{\beta}^{u_{i-1}}$ . This implies that by replacing  $X_{i-1,l}$  in the definition of the cost  $w_{t,i-1,j,k}^{\alpha\gamma\beta}$  by  $X_{il} + \bar{X}_{i-1}$ , the cost  $w_{t,i-1,j,k}^{\alpha\gamma\beta}$  can be computed from  $w_{tijk}^{\alpha\gamma\beta}$  independently of period  $l$  by using the observation above. Therefore, for all  $1 \leq i \leq T$  and given  $k, t, j$ , with  $i \leq k \leq j \leq T$  and  $1 \leq t \leq j$ , for  $\alpha \in \{0, u_{i-1}\}$ ,  $\gamma \in \{0, u_{l*}\}$  and  $\beta \in \{0, u_j\}$ , the cost  $w_{tijk}^{\alpha\gamma\beta}$  can be done in  $\mathcal{O}(T)$  using the following equations:

$$(i) \quad w_{t,i-1,j,k}^{0\gamma\beta} = +\infty \text{ for any value of } w_{tijk}^{0\gamma\beta}$$

$$(ii) \quad w_{t,i-1,j,k}^{u_{i-2}\gamma\beta} = \begin{cases} w_{tijk}^{u_{i-1}\gamma\beta} + h_{i-2}^R u_{i-2} + (p_k^R - \sum_{l=i-1}^{k-1} h_l^R) \bar{X}_{i-1} + \sum_{l=t}^{k-1} h_l^S \bar{X}_{i-1}, \\ \quad \text{if } u_{i-2} > d_{i-1,k-1} \text{ and } d_{il^*} + \beta > u_{i-1} \\ w_{tijk}^{u_{i-1}\gamma\beta} + f_k^R + h_{i-2}^R u_{i-2} + (p_k^R - \sum_{l=i-1}^{k-1} h_l^R) \bar{X}_{i-1} + \sum_{l=t}^{k-1} h_l^S \bar{X}_{i-1}, \\ \quad \text{if } u_{i-2} > d_{i-1,k-1} \text{ and } d_{il^*} + \beta = u_{i-1} \\ + \infty, \quad \text{otherwise} \end{cases}$$

where  $l^* = \operatorname{argmin} w_{tijk}^{u_{i-1}\gamma\beta}$ .

Consequently, for all periods  $i, k, t, j$  such that  $1 \leq i \leq k \leq j \leq T$  and  $1 \leq t \leq j$ , the cost  $w_{tijk}^{\alpha\gamma\beta}$  can be computed in  $\mathcal{O}(T^4)$ . This implies that the algorithm which solves the 2ULS-IB<sub>R</sub> problem runs in  $\mathcal{O}(T^4)$ .

#### 4 Complexity analysis of the 2ULS-IB<sub>S</sub> problem

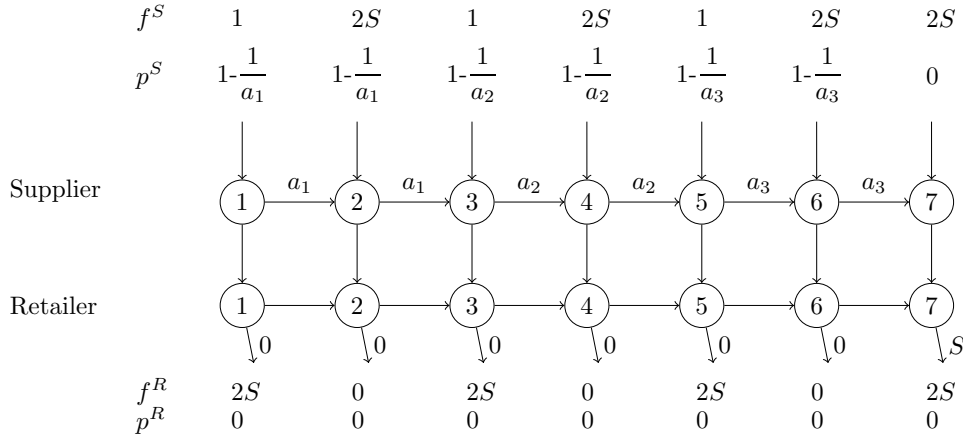
Jaruphongsas *et al.* [5] propose a polynomial time algorithm to solve the 2ULS-IB<sub>S</sub> problem with demand time window constraints and stationary inventory bounds. They consider that  $h^S \leq h^R$  and that the fixed ordering cost and the unit ordering cost are decreasing. These specific costs make the problem solvable in polynomial time. In this section, we consider the 2ULS-IB<sub>S</sub> problem under a general cost structure and we prove that this problem is NP-hard.

**Theorem 1** *The 2ULS-IB<sub>S</sub> problem is NP-hard.*

**Proof.** We prove that the 2ULS-IB<sub>S</sub> problem is NP-hard through a reduction from the subset sum problem, which is an NP-complete problem [3]. An instance of the subset sum problem is given by an integer  $S$  and a set  $\mathcal{S}$  of  $n$  integers  $(a_1, \dots, a_n)$ . The question is: does there exist a subset  $\mathcal{A} \subseteq \mathcal{S}$  such that  $\sum_{a_i \in \mathcal{A}} a_i = S$ ? We transform an instance of the subset sum problem into an instance of the 2ULS-IB<sub>S</sub> problem in the following way:

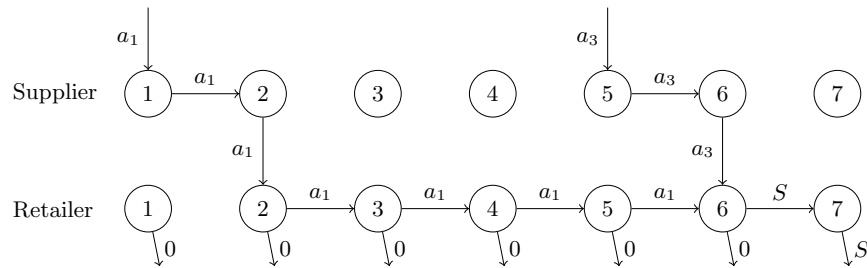
- $T = 2n + 1$ . Let us denote by  $\mathcal{T}_1$  (resp.  $\mathcal{T}_2$ ) the set of odd (resp. even) periods in the set  $\{1, \dots, 2n\}$ .
- $d_t = 0$  for all  $t \in \mathcal{T}_1 \cup \mathcal{T}_2$ ,  $d_T = S$
- $f_t^S = 1$  for all  $t \in \mathcal{T}_1$ ,  $f_t^S = 2S$  for all  $t \in \mathcal{T}_2 \cup \{T\}$   
 $f_t^R = 2S$  for all  $t \in \mathcal{T}_1 \cup \{T\}$ ,  $f_t^R = 0$  for all  $t \in \mathcal{T}_2$
- $h_t^S = h_t^R = 0$  for all  $t \in \{1, \dots, T\}$
- $p_t^R = 0$  for all  $t \in \{1, \dots, T\}$   
 $p_t^S = 1 - 1/a_{\lceil \frac{t}{2} \rceil}$  for all  $t \in \mathcal{T}_1 \cup \mathcal{T}_2$ ,  $p_T^S = 0$
- $u_t^S = a_{\lceil \frac{t}{2} \rceil}$  for all  $t \in \mathcal{T}_1 \cup \mathcal{T}_2$ ,  $u_T^S = 0$

A representation of this instance is given in Figure 5. The fixed ordering costs and the unit ordering costs of the supplier (resp. retailer) are indicated at the top (resp. bottom). At the supplier level, the quantities on the horizontal edges represent the inventory bounds.

Figure 5: Instance  $\mathcal{A}$  of the 2ULS-IB $_S$  problem in the proof of Theorem 1 with  $n = 3$ ,  $\mathcal{S} = \{a_1, a_2, a_3\}$ 

**Observation 3** First of all, note that if we order  $x_t^S = a_{\lceil \frac{t}{2} \rceil}$  at period  $t \in \mathcal{T}_1$  then the total ordering cost is equal to  $f_t^S + p_t^S x_t^S = 1 + (1 - 1/a_{\lceil \frac{t}{2} \rceil})a_{\lceil \frac{t}{2} \rceil}$  which is exactly equal to  $x_t^S$  (in this case, the average cost of ordering one unit is equal to 1). If  $x_t^S < a_{\lceil \frac{t}{2} \rceil}$  at period  $t \in \mathcal{T}_1$ , then we have that the total ordering cost  $f_t^S + p_t^S x_t^S = 1 + (1 - 1/a_{\lceil \frac{t}{2} \rceil})x_t^S = x_t^S + 1 - x_t^S/a_{\lceil \frac{t}{2} \rceil} > x_t^S$  (in this case, the average cost of ordering one unit is larger than 1). From this observation, let us prove that there exists a solution for the 2ULS-IB $_S$  problem of cost at most  $S$  if and only if there exists a solution for the subset sum problem.

Assume that there exists a solution  $\mathcal{A}$  of the subset sum problem. The following solution for the 2ULS-IB $_S$  problem is of cost at most  $S$ : for each element  $a_i$  in the set  $\mathcal{A}$ , the supplier orders a quantity  $a_i$  at period  $t = 2i - 1$  and store it until period  $t + 1$  (see Figure 6). The inventory bound is not exceeded since it is exactly equal to  $a_i$ . From the observation above, the cost of ordering  $a_i$  units for each  $a_i \in \mathcal{I}$  at the supplier level is equal to  $a_i$ . Since  $\sum_{a_i \in \mathcal{I}} a_i = S$ , the total cost at the supplier level is  $S$ . At period  $t = 2i$ , the retailer orders all the units and store them until period  $T$ . Since  $f_t^R = 0$  for all  $t \in \mathcal{T}_2$  and  $h_t^R = p_t^R = 0$  for all  $t \in \{1, \dots, T\}$ , the total cost at the retailer level is equal to 0. So, there exists a solution for the 2ULS-IB $_S$  problem of cost  $S$ .

Figure 6: Solution for the 2ULS-IB $_S$  problem in the proof of Theorem 1 with  $n = 3$ ,  $\mathcal{S} = \{a_1, a_2, a_3\}$  and  $a_1 + a_3 = S$ 

Assume that there exists a solution for the 2ULS-IB $_S$  problem with a cost of at most  $S$ . Since  $f_t^S = 2S$  for all  $t \in \mathcal{T}_2$ , the supplier has to order at period  $t \in \mathcal{T}_1$ , otherwise the cost will exceed  $S$ . Likewise, since  $f_t^R = 2S$  for all  $t \in \mathcal{T}_1$ , the retailer has to order at period  $t \in \mathcal{T}_2$ . In order to not exceed the inventory bounds, the supplier can store at most  $u_t^S = a_{\lceil \frac{t}{2} \rceil}$  units from period  $t$  to period  $t + 1$ . Thus, the quantity

ordered by the supplier at period  $t \in \mathcal{T}_1$  is at most  $a_{\lceil \frac{t}{2} \rceil}$ . At period  $t \in \mathcal{T}_2$ , the retailer orders the units in the supplier's inventory and stores them until period  $T$  with a cost equal to 0. From Observation 1, if the supplier orders at period  $t$ , then  $x_t^S = a_{\lceil \frac{t}{2} \rceil}$  (this is the only way to order one unit with a cost of at most 1 so that the total cost is at most  $S$ ). Thus,  $S = \sum_{t \in \mathcal{T}} a_{\lceil \frac{t}{2} \rceil}$  where  $\mathcal{T}$  is the set of periods where the supplier orders. This implies that there exists a solution to the subset sum problem.  $\square$

## 5 Pseudo-polynomial algorithm for the 2ULS-IB<sub>SR</sub> problem

We have proved that the 2ULS-IB<sub>S</sub> problem is NP-hard. By setting  $u_t^R = \sum_{t=1}^T d_t$ , we can transform an instance of the 2ULS-IB<sub>S</sub> problem into an instance of the 2ULS-IB<sub>SR</sub> problem. Thus, the 2ULS-IB<sub>SR</sub> problem is as hard as the 2ULS-IB<sub>S</sub> problem. In this section, we describe a pseudo-polynomial dynamic programming algorithm to solve the 2ULS-IB<sub>SR</sub> problem which proves that this problem is not strongly NP-hard. The principle of the algorithm is to consider all the possible values of the inventory quantity  $s_t^S \in \{0, 1, \dots, u_t^S\}$  (resp.  $s_t^R \in \{0, 1, \dots, u_t^R\}$ ) at the supplier (resp. retailer) level. Notice that, the ZIO property does not hold both at the supplier and the retailer levels for the 2ULS-IB<sub>SR</sub> problem.

### Computing the total cost of satisfying the demands $d_{tT}$

We assume that  $s_{i-1}^S$  (resp.  $s_{t-1}^R$ ) is the inventory quantity available at the end of period  $i-1$  (resp.  $t-1$ ) at the supplier (resp. retailer) level. In order to define  $H_{it}(s_{i-1}^S, s_{t-1}^R)$ , the optimal cost of satisfying the demands  $d_{tT}$ , we first need to introduce the optimal cost of ordering at the supplier level and the optimal cost of ordering at the retailer level.

*Definition of the cost  $W_{it}(s_{i-1}^S, s_{t-1}^R, s, X^S)$  for ordering at period  $i$  and carrying  $s_i^S = s$  at the supplier level*  
We assume that the retailer is carrying a quantity  $s_{t-1}^R$  in inventory at period  $t-1$ . Moreover, the supplier is ordering a quantity  $X^S$  at period  $i$  and is carrying a quantity  $s_{i-1}^S$  in inventory at period  $i-1$  and a quantity  $s$  at period  $i$ . The minimum cost of satisfying the demands  $d_{tT}$  under these assumptions is given by  $W_{it}(s_{i-1}^S, s_{t-1}^R, s, X^S)$ .

*Definition of the cost  $V_{tk}(s_{t-1}^S, s_{t-1}^R, \bar{s})$  for ordering at period  $t$  and carrying  $s_k^R = \bar{s}$  ( $t \leq k$ ) at the retailer level*

We assume that at the retailer level  $s_k^R = \bar{s}$ , a quantity  $X^R = d_{tk} + \bar{s} - s_{t-1}^R$  is ordered at period  $t$  if  $X^R > 0$  in order to satisfy  $d_{tk}$ , with  $t \leq k \leq T$ . The only ordering period between periods  $t$  and  $k$  at the retailer level is  $t$ . The demands  $d_{k+1,T}$  are satisfied with an order occurring after period  $k$ . These assumptions are illustrated in Figure 7. We define  $V_{tk}(s_{t-1}^S, s_{t-1}^R, \bar{s})$  as the minimum cost of satisfying the demands  $d_{tT}$ .

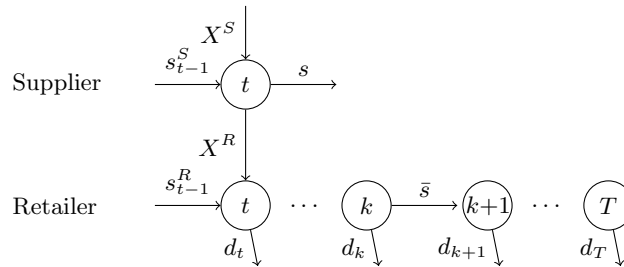


Figure 7: Illustration of the cost  $V_{tk}(s_{t-1}^S, s_{t-1}^R, \bar{s})$

In Figure 7, we can see that in order to satisfy the demands  $d_{tk}$  ( $t \leq k$ ), a quantity  $X^R = d_{tk} + \bar{s} - s_{t-1}^R$  is ordered at period  $t$  at the retailer level, assuming that  $s_k^R = \bar{s}$ . At the supplier level, a quantity

$X^S = X^R + s - s_{t-1}^S$  can be ordered at period  $t$  where  $s_t^S = s \in \{s_{t-1}^S - X^R, \dots, M_{t,k+1}^S\}$  and  $M_{t,k+1}^S = \min(u_t^S, d_{k+1,T})$ .

*Definition of the cost  $H_{it}(s_{i-1}^S, s_{t-1}^R)$*

Let  $i, t$  be two periods such that  $1 \leq i \leq t \leq T$ . We assume that  $s_{i-1}^S \in \{0, 1, \dots, u_{i-1}^S\}$  (resp.  $s_{t-1}^R \in \{0, 1, \dots, u_{t-1}^R\}$ ) is the inventory quantity available at the end of period  $i-1$  (resp.  $t-1$ ) at the supplier (resp. retailer) level. Let us define  $H_{it}(s_{i-1}^S, s_{t-1}^R)$  as the minimum cost of satisfying the demands  $d_{iT}$ .

The following cases will be considered to compute  $H_{it}(s_{i-1}^S, s_{t-1}^R)$ . If there is no ordering at period  $t$  at the retailer level (it is the case if  $i < t$ ), then the supplier holds a quantity  $s \geq s_{i-1}^S$  from period  $i$  to  $i+1$ , and an order can occur at the supplier level. If  $i = t$ , then at the retailer level it is possible to have an ordering at period  $t$  in order to satisfy the demands  $d_{tk}$  ( $t \leq k$ ). The quantity  $\bar{s}$  is the number of units that the retailer holds in the inventory at period  $k$ . The values of  $H_{it}(s_{i-1}^S, s_{t-1}^R)$  can be computed using the following equations:

$$H_{it}(s_{i-1}^S, s_{t-1}^R) = \begin{cases} \min_{s \in \{s_{i-1}^S, \dots, M_{it}^S\}} \{W_{it}(s_{i-1}^S, s_{t-1}^R, s, s - s_{i-1}^S)\}, & \text{if } i < t \\ \min_{t \leq k \leq T} \left\{ \min_{\bar{s} \in \{s_{t-1}^R - d_{tk}, \dots, M_k^R\}} \{V_{tk}(s_{t-1}^S, s_{t-1}^R, \bar{s})\} \right\}, & \text{if } i = t \\ +\infty, & \text{otherwise} \end{cases}$$

where  $M_{it}^S = \min(u_i^S, d_{iT})$  and  $M_k^R = \min(u_k^R, d_{kT})$ .

### Computing the cost of ordering at the supplier level

*Definition of the cost  $C_i^S(X^S)$  for ordering units at the supplier level*

Let  $C_i^S(X^S)$  be the cost of ordering a quantity  $X^S$  at period  $i$  at the supplier level. We have that:

$$C_i^S(X^S) = \begin{cases} f_i^S + p_i^S X^S, & \text{if } X^S > 0 \\ 0, & \text{otherwise} \end{cases}$$

The minimum cost  $W_{it}(s_{i-1}^S, s_{t-1}^R, s, X^S)$  of ordering at period  $i$  at the supplier level and carrying  $s$  units at period  $i$  for satisfying the demands  $d_{iT}$  is given by:

$$W_{it}(s_{i-1}^S, s_{t-1}^R, s, X^S) = \begin{cases} h_i^S s + C_i^S(X^S) + H_{i+1,t}(s, s_{t-1}^R), & \text{if } s \leq u_i^S \text{ and } s_{i-1}^S + X^S + s_{t-1}^R \leq d_{iT} \\ +\infty, & \text{otherwise} \end{cases}$$

At the supplier level, if the inventory quantity  $s$  exceeds the inventory bound  $u_i^S$ , then the cost  $W_{it}(s_{i-1}^S, s_{t-1}^R, s, X^S)$  is equal to  $+\infty$ .

### Computing the cost of ordering at the retailer level

*Definition of the cost  $C_{tk}^R(X^R, s_{t-1}^R)$  for ordering units at the retailer level*

Satisfying the demands  $d_{tk}$  at the retailer level by ordering a quantity  $X^R$  at period  $t$  and holding  $s_{t-1}^R$  at period  $t-1$  induces a cost  $C_{tk}^R(X^R, s_{t-1}^R)$  given by:

$$C_{tk}^R(X^R, s_{t-1}^R) = \begin{cases} \sum_{l=t}^k h_l^R (s_{t-1}^R - d_{tl}), & \text{if } X^R = 0 \\ f_t^R + p_t^R X^R + \sum_{l=t}^k h_l^R (X^R + s_{t-1}^R - d_{tl}), & \text{otherwise} \end{cases}$$

Then, the cost  $V_{tk}(s_{t-1}^S, s_{t-1}^R, \bar{s})$  of satisfying the demands  $d_{tT}$  is computed as follows:

$$V_{tk}(s_{t-1}^S, s_{t-1}^R, \bar{s}) = \begin{cases} C_{tk}^R(X^R, s_{t-1}^R) + \min_{s \in \{s_{t-1}^S - X^R, \dots, M_{t,k+1}^S\}} \{W_{t,k+1}(s_{t-1}^S, \bar{s}, s, X^S)\}, \\ \quad \text{if } s_{t-1}^R + X^R - d_t \leq u_t^R \\ +\infty, \text{ otherwise} \end{cases}$$

At the retailer level, if the inventory quantity at period  $t$ , defined by  $s_{t-1}^R + X^R - d_t$ , exceeds the inventory bound  $u_t^R$  then the cost  $V_{tk}(s_{t-1}^S, s_{t-1}^R, \bar{s})$  is equal to  $+\infty$ .

### Optimal cost

The optimal cost of satisfying the demands  $d_{1T}$  assuming that  $s_0^R = s_0^S = 0$  is given by  $H_{11}(0,0)$ . We initialize the recursion by setting  $H_{t,T+1}(s_{t-1}^S, s_{t-1}^R) = 0$  for all  $t \in \{1, \dots, T+1\}$  and for all the values  $s_{t-1}^R$  and  $s_{t-1}^S$  ensuring feasibility.

### Complexity analysis

The cost  $W_{it}(s_{i-1}^S, s_{i-1}^R, s, X^S)$  is computed in  $\mathcal{O}(u_{i-1}^S u_{i-1}^R M_{it}^S)$  for each pair of periods  $(i, t)$ . Computing  $V_{tk}(s_{t-1}^S, s_{t-1}^R, \bar{s})$  can be done in  $\mathcal{O}(u_{t-1}^S u_{t-1}^R M_k^R M_{t,k+1}^S)$  for each pair of periods  $(k, t)$ .

Finally, it takes  $\mathcal{O}(\sum_{i=1}^T (\sum_{t=i+1}^T (u_{i-1}^S u_{t-1}^R M_{it}^S) + \sum_{k=i}^T (u_{i-1}^S u_{i-1}^R M_k^R M_{i,k+1}^S)))$  to compute the cost  $H_{it}(s_{i-1}^S, s_{i-1}^R)$  for all the pairs of periods  $(i, t)$ . This bound constitutes the complexity of the dynamic programming algorithm which is pseudo-polynomial, implying that the 2ULS-IB<sub>SR</sub> problem is not strongly NP-hard.

In the next section, we will consider the 2ULS problems with inventory bounds assuming that the no lot-splitting constraint holds.

## 6 Analysis of lot-sizing problems with the NLS constraint

In this section, we consider that at the retailer level the demand lot-splitting is not allowed. We will denote this constraint as NLS (No Lot-Splitting). The NLS constraint imposes that each demand  $d_t$ , for  $1 \leq t \leq T$ , is satisfied by one single ordering. More formally, we note  $x_{kt}^R \geq 0$  the quantity of demand  $d_t$  which is ordered at period  $k$  to satisfy a demand  $d_t$  at the retailer level. We have  $\sum_{i=1}^t x_{it}^R = d_t$ .

**Definition 5 (NLS constraint)** *A demand  $d_t$ , with  $1 \leq t \leq T$ , fulfills the NLS constraint in an ordering plan  $x^R$  if there does not exist two periods  $l$  and  $k$  with  $l < k \leq t$  such that  $x_{lt}^R > 0$  and  $x_{kt}^R > 0$ . An ordering plan  $x^R$  fulfills the NLS constraint if all the demands  $d_t$ ,  $t \in \{1, \dots, T\}$ , fulfill the NLS constraint in  $x^R$ .*

Before studying the complexity of 2ULS problems with inventory bounds problems and the NLS constraint, it is interesting to analyze the complexity of the single level problem with NLS constraint, that we denote by ULS-IB-NLS.

### 6.1 Complexity of the ULS-IB-NLS problem

We consider  $T$  periods  $\{1, \dots, T\}$ . In the ULS-IB-NLS problem, ordering units at period  $t$  induces a fixed ordering cost  $f_t$  and a unit ordering cost  $p_t$ . Carrying units from period  $t$  to period  $t+1$  induces a holding cost  $h_t$ . The total cost is given by the sum of the ordering and holding costs. The aim is to determine an ordering plan which satisfies the demands and which minimizes the total cost. We denote by  $x_t$  the ordering quantity at period  $t$ ,  $s_t$  the inventory quantity at the end of period  $t$  and  $y_t$  the binary (setup) variable which is equal to 1 if there is an order at period  $t$  and 0 otherwise.



We say that the inventory bound is *stationary* if  $u_t$  is constant throughout the planning horizon. We prove that the ULS-IB-NLS problem is strongly NP-hard.

**Theorem 2** *The ULS-IB-NLS problem is strongly NP-hard, even if the inventory bound is stationary.*

**Proof.** We show that the 3-Partition problem, which is strongly NP-hard [3], can be reduced to the ULS-IB-NLS problem in polynomial time. Recall that an instance of the 3-Partition problem is given by an integer  $b$  and  $3m$  integers  $(a_1, \dots, a_{3m})$  such that  $\sum_{i=1}^{3m} a_i = mb$  and  $b/4 < a_i < b/2$  for all  $i \in \{1, \dots, 3m\}$ . The question is: does there exist a partition  $A_1 \cup \dots \cup A_m$  of  $\{1, \dots, 3m\}$  such that  $\sum_{i \in A_j} a_i = b$  for all  $j \in \{1, \dots, m\}$ .

We transform an instance of the 3-Partition problem into an instance of the ULS-IB-NLS problem in the following way:

- $T = 5m$  periods. Let us note  $\mathcal{T}_1$  (resp.  $\mathcal{T}_2$ ) the set of odd (resp. even) periods in the set  $\{1, \dots, 2m\}$ .
- $d_t = 0$  for all  $t \in \mathcal{T}_1$   
 $d_t = (m - t/2)b$  for all  $t \in \mathcal{T}_2$   
 $d_t = a_{t-2m}$  for all  $t \in \{2m+1, \dots, T\}$
- $f_t = 0$  for all  $t \in \mathcal{T}_1$   
 $f_t = 1$  for all  $t \in \mathcal{T}_2 \cup \{2m+1, \dots, T\}$
- $h_t = p_t = 0$  for all  $t \in \{1, \dots, T\}$
- $u_t = mb$  for all  $t \in \{1, \dots, T\}$

The instance is illustrated in Figure 8. The fixed ordering costs are indicated at the top of each period. The inventory bounds are represented on the horizontal edges.

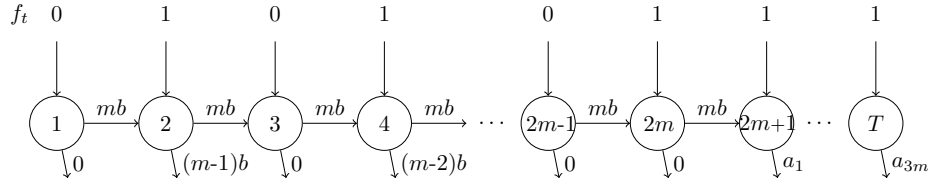


Figure 8: Instance of the ULS-IB-NLS problem in the proof of Theorem 2

Let us show that there exists a solution to the ULS-IB-NLS problem of cost at most 0 if and only if there exists a solution to the 3-Partition problem.

Assume that there exists a solution  $(A_1, \dots, A_m)$  of the 3-Partition problem. The cost of the following solution of the ULS-IB-NLS problem is 0: at each period  $t \in \mathcal{T}_1$ , we order  $x_t = \sum_{i \in A_{(t+1)/2}} a_i + d_{t+1} = b + b(m - \frac{t+1}{2})$  units. Since the ordering cost is equal to 0 for all  $t \in \mathcal{T}_1$ , it costs 0 to order these units. At each period  $t \in \mathcal{T}_2$ , the demand  $d_t$  is satisfied and  $b$  units are stored which implies that there is exactly  $s_t = \frac{t}{2}b$  units in stock at the end of period  $t$ . At each period  $t \in \mathcal{T}_1$ , we store exactly a quantity  $s_{t-1} + x_t = \frac{t-1}{2}b + (m - \frac{t-1}{2})b = mb$  and the inventory bound  $u_t$  is not exceeded. Each demand  $d_t$  for all  $t < 2m$  is satisfied and there is  $mb$  units in stock at period  $2m$  for satisfying the demands at period  $\{2m+1, \dots, T\}$ . Since there is no holding cost, the cost of this solution is 0. Note that this solution fulfills the NLS constraint since each demand is satisfied by a single order.

Assume now that there exists a solution to the ULS-IB-NLS problem of cost at most 0 (see Figure 9).

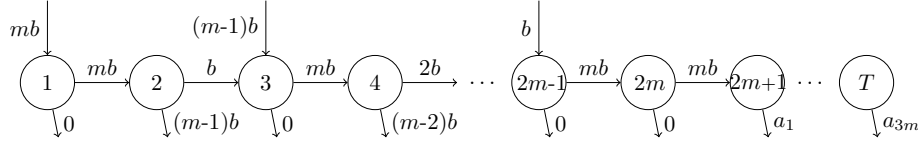


Figure 9: Solution for the ULS-IB-NLS problem in the proof of Theorem 2

Since the fixed ordering cost is equal to 1 for all  $t \in \mathcal{T}_2 \cup \{2m+1, \dots, T\}$ , we cannot order at these periods. Thus, all orders are set at period  $t \in \mathcal{T}_1$ . Since for each period  $t \in \mathcal{T}_2$ ,  $d_t = (m-t/2)b$ , and since the inventory bound is  $mb$ , at most  $\frac{t}{2}b$  units can be stored from period  $t \in \mathcal{T}_2$  to a period in  $\mathcal{T}_1$ . Since  $2m$  units have to be available at period  $2m$  (otherwise the cost will be greater than 0), then  $\frac{t}{2}b$  units have to be stored from period  $t \in \mathcal{T}_2$  to period  $t+1$ . So, we have to order  $b$  units at each period  $t \in \mathcal{T}_1$  for satisfying the demands  $d_{2m+1,T}$ . Assuming the NLS constraint, each demand  $d_t$  for all  $t \in \{2m+1, \dots, T\}$  is satisfied by a single ordering period at  $t \in \mathcal{T}_1$ . So, there is a partition of the periods  $\{2m+1, \dots, T\}$  into  $m$  sets  $(A_1, \dots, A_m)$  such that  $\sum_{i \in A_j} d_i = b$  for all  $j \in \{1, \dots, m\}$ . Since each demand  $d_t$  for all  $t \in \{2m+1, \dots, T\}$  corresponds to an integer of  $(a_1, \dots, a_{3m})$ , this means that there exists a solution to the 3-Partition problem.  $\square$

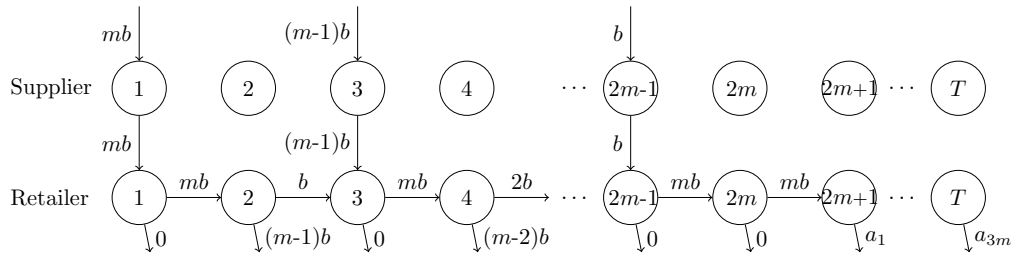
Note that ULS-IB problem can be solved in polynomial time [2], [7]. Theorem 2 shows that adding the NLS constraint to this problem makes it strongly NP-hard.

Let us now prove that the 2ULS-IB<sub>R</sub> and the 2ULS-IB<sub>S</sub> problems with the NLS constraint, denoted by 2ULS-IB<sub>R</sub>-NLS and 2ULS-IB<sub>S</sub>-NLS respectively, are strongly NP-hard.

## 6.2 Complexity of the 2ULS-IB<sub>R</sub>-NLS problem

**Theorem 3** *The 2ULS-IB<sub>R</sub>-NLS problem is strongly NP-hard, even if the inventory bound is stationary.*

**Proof.** We do a reduction from the ULS-IB-NLS problem, that is strongly NP-hard, as shown by Theorem 2. We transform an instance of the ULS-IB-NLS problem into the following instance of the 2ULS-IB<sub>R</sub>-NLS problem. The costs of the retailer are the ones of the ULS-IB-NLS problem, *i.e.*  $u_t^R = u_t, f_t^R = f_t, p_t^R = p_t$  and  $h_t^R = h_t$  for all  $t \in \{1, \dots, T\}$ . The supplier costs are given by  $f_t^S = h_t^S = p_t^S = 0$  for all  $t \in \{1, \dots, T\}$ . The demands are the same than the ones of the ULS-IB-NLS problem. Since all the supplier's costs are 0, the cost of an optimal solution for the ULS-IB-NLS problem is equal to the optimal cost of its corresponding 2ULS-IB<sub>R</sub>-NLS instance (see Figure 10).

Figure 10: Solution for the 2ULS-IB<sub>R</sub>-NLS problem in the proof of Theorem 3

By Theorem 2, the 2ULS-IB<sub>R</sub>-NLS problem is also strongly NP-hard.  $\square$

### 6.3 Complexity of the 2ULS-IB<sub>S</sub>-NLS problem

Jaruphongsa *et al.* [5] prove that the 2ULS-IB<sub>S</sub>-NLS problem with demand time window constraints is weakly NP-hard. We show that this problem is also weakly NP-hard without demand time window constraints, and that it is even strongly NP-hard.

**Theorem 4** *The 2ULS-IB<sub>S</sub>-NLS is strongly NP-hard, even if the inventory bound is stationary.*

**Proof.** As in the proof of Theorem 3, we do a reduction from the ULS-IB-NLS problem, which is strongly NP-hard, as shown in Theorem 2. We transform an instance of the ULS-IB-NLS problem into the following instance of problem 2ULS-IB<sub>S</sub>-NLS. The supplier's costs are the ones of the ULS-IB-NLS problem, *i.e.*  $u_t^S = u_t$ ,  $f_t^S = f_t$ ,  $p_t^S = p_t$  and  $h_t^S = h_t$  for all  $t \in \{1, \dots, T\}$ . The retailer's costs are given by  $f_t^R = p_t^R = 0$  for all  $t \in \{1, \dots, T\}$  and, for all  $t \in \{1, \dots, T\}$ ,  $h_t^R = M$ , where  $M$  is a large number (we can fix  $M = \sum_{t=1}^T (h_t + p_t)$ ). By this way, in an optimal solution of the 2ULS-IB<sub>S</sub>-NLS problem, no quantity will be stored at the retailer level. The demands are the same than the ones of the ULS-IB-NLS problem. Since  $f_t^R = p_t^R = 0$ , the cost of an optimal solution of the 2ULS-IB<sub>S</sub>-NLS problem is equal to the optimal cost of its corresponding ULS-IB-NLS problem. Figure 11 illustrates such a solution.

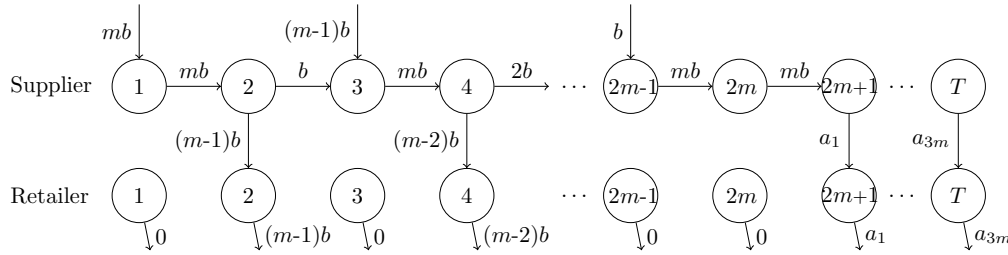


Figure 11: Solution for the 2ULS-IB<sub>S</sub>-NLS problem in the proof of Theorem 4

Therefore, by Theorem 2, the 2ULS-IB<sub>S</sub>-NLS problem is also strongly NP-hard.  $\square$

### 6.4 Complexity of the 2ULS-IB<sub>SR</sub>-NLS problem

Consider the case where the supplier and the retailer have inventory bounds. We prove that the 2ULS-IB<sub>SR</sub>-NLS problem is strongly NP-hard.

**Theorem 5** *The 2ULS-IB<sub>SR</sub>-NLS problem is strongly NP-hard, even if the inventory bound is stationary.*

**Proof.** The proof of this theorem is the same as the one of Theorem 4 for the 2ULS-IB<sub>S</sub>-NLS problem by adding any inventory bound at the retailer level (in an optimal solution no quantity will be stored at the retailer level).  $\square$

## 7 Conclusion and future work

This paper considers two-level uncapacitated lot-sizing problems with inventory bounds, and provides a complexity analysis of these problems. We present an  $\mathcal{O}(T^4)$  dynamic programming algorithm which solves the problem where the inventory bounds are set at the retailer level. When the inventory bounds are set at the supplier level, we prove that the problem is weakly NP-hard. We also present a pseudo-polynomial dynamic programming algorithm which ensures that this problem is not strongly NP-hard. Considering that lot-splitting is not allowed, we prove that the ULS problem with inventory bounds and the 2ULS problems

where the inventory bounds are set either at the retailer level, or at the supplier level or at both of them are strongly NP-hard.

The following tables summarize the complexity results for 2ULS-IB problems:

Problem	Complexity
ULS-IB	polynomial [2], [7]
2ULS-IB <sub>R</sub>	polynomial (Section 3)
2ULS-IB <sub>S</sub>	polynomial with particular cost structure [5]
	NP-hard(Section 4)
2ULS-IB <sub>SR</sub>	NP-hard (Section 5)

Table 1: Complexity results with lot-splitting

Problem	Complexity
ULS-NLS	strongly NP-hard (Section 6.1)
2ULS-IB <sub>R</sub> -NLS	strongly NP-hard (Section 6.2)
2ULS-IB <sub>S</sub> -NLS	weakly NP-hard with demand time windows [5]
	strongly NP-hard (Section 6.3)
2ULS-IB <sub>SR</sub> -NLS	strongly NP-hard (Section 6.4)

Table 2: Complexity results without lot-splitting

It would be interesting for a future work to improve the running time of the algorithm solving the 2ULS-IB<sub>R</sub> problem. Moreover, the complexity of the 2ULS-IB<sub>S</sub> problem where the inventory bounds of the supplier are stationary is an open problem. Another interesting perspective is to consider that the supplier and the retailer share the same inventory facility. In this case, at each period, the inventory quantity of the supplier plus the one of the retailer cannot exceed a given inventory bound. The lot-sizing problems that have been studied in this paper consider a single item. It would also be interesting to study the case where there are several items. Finally, investigating efficient algorithms to solve the NP-hard 2ULS problems with inventory bounds is also a promising issue for practical applications.

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